J. Fluid Mech. (1973), *vol.* 61, *part* 1, *pp.* 109–127
Printed in Great Britain

Reflexion and transmission at a plane screen with periodically arranged circular or elliptical apertures

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(Received 16 March 1973)

A plane harmonic sound wave is considered to be incident upon a rigid plane screen that contains a periodic rectangular array of circular or elliptical apertures, and a characterization is sought for the reflexion and transmission coefficients of the scattered waves when the relationships

$$\text{aperture dimension} \ll \text{spacing} \ll \text{wavelength}$$

apply. The problem is analysed with the help of an integral equation over a single aperture and, as a consequence of the determination of the leading terms in its asymptotic solution, some prior results for more general (that is, irregular) aperture spacing are confirmed and specific features of the interaction in the periodic arrangement are established. Similar formulations are devised and given attention for the related problems in which (i) the screen is backed by a rigid infinite plane and (ii) the apertures contain rigid pistons capable of executing normal displacements compatible with an assigned and common impedance. A section is devoted to the solution, based on expansion of its kernel, for an integral equation of the first kind with a plane circular or elliptical domain.

1. Introduction

A periodic rectangular array of apertures (having circular or elliptical shape) is located in a rigid plane screen, with the separation between neighbouring units thereof assumed to be large on the scale furnished by the dimensions of an individual aperture. It is desired to study the reflexion and transmission properties of such a screen when irradiated by time-harmonic plane (sound) waves of wavelength large compared with the aperture spacing.

The scattering problem thus posed can, without the aforesaid restrictions on the characteristic lengths, be reduced to a matter of solving a single integral equation of the first kind for a function that describes the normal velocity distribution across a single aperture; although there is no available form of solution encompassing arbitrary relative magnitudes of the various lengths which figure in the problem, an asymptotic solution appropriate to the chosen circumstances, namely

aperture dimensions $\ll$ spacing between neighbouring apertures $\ll$ wavelength,

is feasible. Estimates for the reflexion and transmission coefficients follow from these conditions and both duplicate earlier results obtained by Ffowcs Williams (1972) in the case of non-periodically distributed apertures and also furnish small corrections which depend in a complicated fashion on the twin spacing parameters of the periodic planar array.

The pertinent asymptotic solution, which satisfies a version of the generally valid integral equation suited to the prevailing circumstances, is derived by a direct approach to the integral equation that differs from the one employed by Copson and that is, moreover, equally competent in the cases of circular or elliptical aperture geometry. Specifically, an orthogonal expansion for the reciprocal distance between any two points in the aperture, or kernel of the approximate integral equation, affords a means of solving this equation through development, in terms of the orthogonal family, of the other functions occurring therein. A constant inhomogeneous term of the integral equation alone is relevant here and thus the solution takes an extremely simple form.

The character of the reflexion changes appreciably when the perforated screen is backed by a parallel and infinite rigid plate, and likewise when the apertures contain close-fitting rigid pistons that can move normal to the plane of the screen in accord with a given average impedance. An asymptotic solution for the latter configuration proves to yield agreement with results found by Ffowcs Williams and additionally, supplies a small correction.

The integral equation reformulation of a scattering problem for the perforated screen is given in §2 and its detailed solution is set out in appendix B. Reflexion by a perforated screen with an infinite rigid backing plane and by a periodic distribution of baffled pistons in an otherwise rigid plane are subsequently discussed in §§3 and 4. The Green's functions which enter the respective integral equations of the different problems are explicitly represented and approximated at long wavelengths in appendix A.

2. Reflexion and transmission of plane waves by a periodically perforated screen

A time-periodic plane wave is obliquely incident upon an infinite rigid screen that lies in the plane $z = 0$ and contains a periodic rectangular array of small circular or elliptical apertures with centres at the points $x = md_1$, $y = nd_2$, where m and n are integers (figure 1). The pressure variation of the incident or primary wave is given by

$$p_i e^{-i\omega t} = \exp\{-ik(z\cos\theta + x\sin\theta) - i\omega t\} \tag{2.1}$$

if the wave and screen normals have angular separation θ and the wavenumber $k = \omega/c$ expresses the ratio of the angular frequency ω to the speed of sound c; as the time factor $e^{-i\omega t}$ occurs linearly throughout it will be suppressed in the analysis that follows.

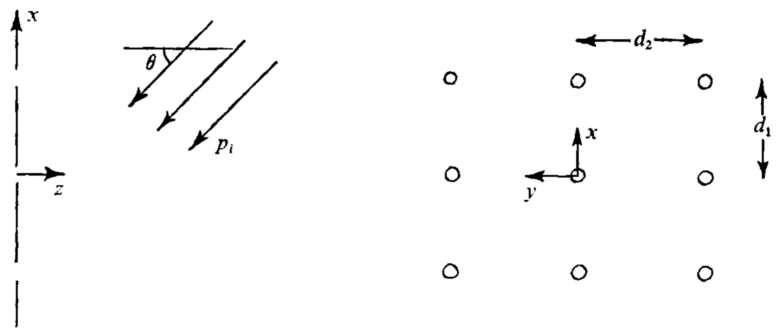

FIGURE 1. The co-ordinate system, incident wave p_i and spacings d_1 and d_2 depicted for the circular aperture configuration.

Our aim is to describe some aspects of the secondary pressure field scattered by the screen when the wavelength $2\pi/k$ is large and the aperture dimension a (radius or semi-major axis for the circular and elliptical shapes) is small compared with the spacing parameters d_1 and d_2. Thus, if $d_<$ and $d_>$ denote the lesser and greater of d_1 and d_2, respectively, the inequalities $kd_> \ll 1$ and $a/d_< \ll 1$ apply. Far from the plane of the screen, in particular, the scattered and primary waves are of the same type, whence

$$p \sim \begin{cases} p_i + R_1 \exp\{-ik(-z\cos\theta + x\sin\theta)\} & \text{as} \quad z \to +\infty, \quad (2.2) \\ T_1 \exp\{-ik(z\cos\theta + x\sin\theta)\} & \text{as} \quad z \to -\infty, \quad (2.3) \end{cases}$$

where R_1 and T_1 are complex constants to be found.

If we express the total pressure fluctuation $p(x, y, z)$ in the form

$$p(x,y,z) = \begin{cases} p_i(x,y,z) + p_i(x,y,-z) + \psi(x,y,z), & z > 0, \\ \psi(x,y,z), & z < 0, \end{cases} \tag{2.4}$$

the antisymmetry of the scattered component $p - p_i$ relative to the plane of the screen implies that $\psi(x, y, z) = -\psi(x, y, -z)$ and consequently attention can be restricted to the half-space $z \geqslant 0$. The function $\psi(x, y, z)$ is there specified by the relations

$$(\nabla^2 + k^2)\psi = 0, \quad z > 0, \tag{2.5}$$

$$\partial\psi/\partial z = 0 \quad \text{when} \quad z = 0 \quad \text{on the rigid screen,} \tag{2.6}$$

$$\psi = -p_i(x, y, 0) \quad \text{when} \quad z = 0 \quad \text{in the apertures,} \tag{2.7}$$

together with a radiation condition stipulating an outgoing-wave behaviour for ψ as $z \to +\infty$ and an edge condition which limits the singularity of $\partial\psi/\partial z$ to one of inverse-square-root nature at the rim of any aperture. In accordance with the spatial periodicity of the incident wave and of the geometrical configuration, we seek a solution of the system (2.5)–(2.7) that has the property

$$\psi(x + md_1, y + nd_2, z) = \exp[-ikd_1 m \sin\theta]\, \psi(x, y, z). \tag{2.8}$$

Although our formulation indicates that the wavenumber is real, mathematical convenience results from assigning it a small positive imaginary part, say

$k = k_1 + ik_2$, where $k_2 \to +0$ ultimately, for such a strategem improves the convergence of sums that occur in the intermediate stages of the analysis.

The boundary-value problem posed above is readily converted to a more succinct form, namely a single relation or integral equation for the function $\partial\psi/\partial z$ over the aperture centred at the origin; this transformation is effected through the use of a Green's function $G(\mathbf{r}, \mathbf{r}')$ such that

$$-4\pi G(\mathbf{r}, \mathbf{r}') = \frac{e^{ikR_+}}{R_+} + \frac{e^{ikR_-}}{R_-}, \quad R_\pm^2 = (x-x')^2 + (y-y')^2 + (z \pm z')^2, \tag{2.9}$$

with a vanishing z' derivative at $z' = 0$. On applying Green's formula to the functions $\psi(\mathbf{r}')$ and $G(\mathbf{r}, \mathbf{r}')$ in the half-space $z' \geqslant 0$, it follows that

$$\psi(x,y,z) = \sum_{m,n=-\infty}^{\infty} \int_{A_{mn}} G(x,y,z;\, x',y',0) \frac{\partial}{\partial z'} \psi(x',y',0)\, dx'\, dy', \quad z > 0, \tag{2.10}$$

where the integrals are extended over all the apertures, A_{mn} designating the one whose centre is located at $x = md_1$, $y = nd_2$. If we define a periodic Green's function, viz.

$$\mathscr{G}(\mathbf{r}, \mathbf{r}') = \sum_{m,n=-\infty}^{\infty} G(x,y,z;\, x'+md_1, y'+nd_2, z') \exp[-ikd_1 m \sin\theta], \tag{2.11}$$

and enforce the corresponding property (2.8) of the regular wave function ψ, the representation (2.10) simplifies to an integral over the single aperture $A \equiv A_{00}$ with centre at the origin and becomes

$$\psi(x,y,z) = \int_A \mathscr{G}(x,y,z;x',y',0) \frac{\partial}{\partial z'} \psi(x',y',0)\, dx'\, dy', \quad z > 0. \tag{2.12}$$

The normal velocity distribution $\partial\psi/\partial z$ across the aperture is fixed by letting the point (x,y,z) tend to any point $(x,y,+0)$ in A and imposing the requirement (2.7), which yields

$$\int_A \mathscr{G}(x,y,0;\, x',y',0) \exp[ik(x-x')\sin\theta]\, V(x',y',)\, dx'\, dy' = -1, \quad (x,y) \text{ in } A, \tag{2.13}$$

with
$$V(x,y) = \exp[ikx\sin\theta]\, \partial\psi(x,y,0)/\partial z. \tag{2.14}$$

As shown in appendix A

$$\mathscr{G}(x,y,z;\, x',y',0) \sim -\frac{i}{kd_1 d_2 \cos\theta} \exp\{ik(z\cos\theta - (x-x')\sin\theta)\}, \quad z \to \infty, \tag{2.15}$$

whence, from (2.12), the wave function assumes the asymptotic form

$$\psi(x,y,z) \sim \frac{-iQ}{kd_1 d_2 \cos\theta} \exp\{ik(z\cos\theta - x\sin\theta)\}, \quad z \to +\infty, \tag{2.16}$$

with the amplitude factor

$$Q = \int_A V(x,y)\, dx\, dy. \tag{2.17}$$

Reference to (2.4) and the fact that ψ is an odd function of z furnishes us, accordingly, with the expressions for the reflexion and transmission coefficients

$$R_1 = 1 - \frac{iQ}{kd_1 d_2 \cos\theta}, \quad T_1 = \frac{iQ}{kd_1 d_2 \cos\theta} \tag{2.18}$$

in terms of the integrated value Q of the aperture distribution $V(x, y)$.

To secure an analytical representation of $V(x, y)$ the basic integral equation (2.13) requires approximation. When $kd_> \ll 1$ and $a/d_< \ll 1$, the characterization (see appendix A)

$$-2\pi\mathscr{G}(x, y, 0;\, x', y', 0) = 1/R + C + o(1) \tag{2.19}$$

is obtained, where

$$R^2 = (x-x')^2 + (y-y')^2$$

and
$$C = \frac{2i\pi}{kd_1 d_2 \cos\theta} + \frac{2}{d_2}\left[\gamma + \log\left(\frac{d_1}{4\pi d_2}\right) + 4\sum_{m,n=1}^{\infty} K_0\left(\frac{2mn\pi d_1}{d_2}\right)\right], \tag{2.20}$$

with the Euler constant $\gamma = 0{\cdot}5771\ldots$, and $K_0(\tau)$ a cylinder function. If the estimate (2.19), which has uniform validity for all (x, y) and (x', y') in the aperture A, is employed in the integral equation and the phase factor set equal to unity, a first approximation $V_1(x, y)$ to the unknown function $V(x, y)$ satisfies the equation

$$\frac{1}{2\pi}\int_A \left(\frac{1}{R} + C\right) V_1(x', y')\, dx'\, dy' = 1, \quad (x, y) \text{ in } A. \tag{2.21}$$

The analysis presented in appendix B establishes that the integral equation

$$\frac{1}{2\pi}\int_A \frac{f(x', y')}{R}\, dx'\, dy' = \alpha, \quad (x, y) \text{ in } A, \tag{2.22}$$

has the explicit solutions

$$f(x, y) = \frac{2\alpha}{\pi a}\left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}}, \quad 0 \leqslant r^2 = x^2 + y^2 \leqslant a^2, \tag{2.23}$$

for a circular aperture A of radius a, and

$$f(x, y) = \frac{\alpha}{bK(\epsilon)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}}, \quad 0 \leqslant \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1, \tag{2.24}$$

for an elliptical aperture A with semi-axes a and b. Here $\epsilon = (1 - b^2/a^2)^{\frac{1}{2}}$ is the eccentricity and

$$K(\epsilon) = \int_0^{\frac{1}{2}\pi} (1 - \epsilon^2 \sin^2\phi)^{-\frac{1}{2}}\, d\phi$$

designates a complete elliptic integral. A comparison of (2.21) and (2.22) reveals the connexions

$$V_1(x, y) = f(x, y), \quad \alpha = 1 - \frac{C}{2\pi} Q_1 = 1 - \frac{C}{2\pi}\int_A V_1(x, y)\, dx\, dy,$$

whence
$$V_1(x, y) = \frac{2}{\pi}\left(1 - \frac{C}{2\pi} Q_1\right)(a^2 - r^2)^{-\frac{1}{2}} \tag{2.25}$$

in the circular case. On integrating (2.25) over the aperture we find that

$$Q_1 = 4a\pi/(\pi + 2aC) \tag{2.26}$$

and therefore, from (2.18) and (2.20), the corresponding forms of the reflexion and transmission coefficients are

$$R_1 = 1 - T_1 = \frac{1 + aF(d_1, d_2)}{1 + (4ia/kd_1 d_2 \cos\theta) + aF(d_1, d_2)} + o(a/d_<), \tag{2.27}$$

where

$$F(d_1, d_2) = F(d_2, d_1) = \frac{4}{\pi d_2}\left[\gamma + \log\left(\frac{d_1}{4\pi d_2}\right) + 4\sum_{m,n=1}^{\infty} K_0\left(\frac{2mn\pi d_1}{d_2}\right)\right]. \tag{2.28}$$

The version of (2.27) that holds when the small term $aF(d_1, d_2)$ is omitted confirms the result found by Ffowcs Williams (1972) for a plane in which the apertures are densely clustered, with an average number N per unit area; for the contrast of formulae, a value $N = (d_1 d_2)^{-1}$ is assigned in the periodic arrangement contemplated here. It would appear from the particular form of the structure factor $F(d_1, d_2)$, which is linked specifically to the rectangular arrangement of apertures, that any improvement on Ffowcs Williams' approximation for more general aperture distributions would involve a very complicated dependence on the spacing parameters.

The estimate (2.27) for the reflexion and transmission coefficients of a perforated or porous screen is applicable whatever the magnitude of the ratio $4ia/(kd_1 d_2 \cos\theta)$ that occurs therein. If this quantity (and the effective porosity of the screen) is small, then an incident plane wave evidently undergoes nearly total reflexion; and if $4ia/(kd_1 d_2 \cos\theta) \gg 1$, corresponding to a large number of apertures per unit area or an incident wave close to the grazing direction, there is nearly complete transmission. Physical interpretations of both limiting cases, in terms of source and dipole layers, are discussed by Ffowcs Williams (1972).

An effective impedance Z, of the screen, may be adopted in place of the reflexion coefficient R as a measure of its scattering quality and, in conjunction with the relationship

$$R = (Z\cos\theta - \rho c)/(Z\cos\theta + \rho c),$$

where ρ is the air density, there follows a small wavenumber expansion

$$Z/\rho c = \sec\theta - i(kd_1 d_2/2a)(1 + aF) + o(kd_>). \tag{2.29}$$

For apertures of elliptical shape the counterparts of (2.25) and (2.26), namely

$$V_1(x, y) = \left(1 - \frac{cQ_1}{2\pi}\right)\frac{1}{bK(\epsilon)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} \tag{2.30}$$

and

$$Q_1 = 2\pi a/(K(\epsilon) + aC) = 4\bar{a}\pi/(\pi + 2\bar{a}C), \tag{2.31}$$

make it evident that the reflexion coefficient R_1 in (2.26) carries over if the circular radius a appearing therein is replaced by an equivalent magnitude

$$\bar{a} = \pi a/2K(\epsilon), \tag{2.32}$$

which varies with the eccentricity in a definite manner. It is necessary to assume, of course, that the major axes of the ellipses are parallel, so as to preserve the periodicity of the configuration, though the above analysis remains valid for any orientation of the ellipses relative to the x and y axes in the plane of the screen.

To characterize the excitation in the neighbourhood of the apertures we consider the circular shape for simplicity and denote by $F(\mathbf{x})$ the harmonic function that vanishes at infinity and has the normal derivative

$$\frac{\partial F}{\partial z} = \begin{cases} (2/\pi)(a^2-r^2)^{-\frac{1}{2}}, & r < a, \\ 0, & r > a, \end{cases} \tag{2.33}$$

in the plane $z = 0$. Reference to the expression (2.25) for the normal aperture velocity reveals that the pressure distribution ψ near A has (apart from incident and reflected wave components) a locally incompressible nature, with the representation

$$\psi \sim S_1 F(\mathbf{x}), \tag{2.34}$$

where the scaling constant S_1, which accounts for the interactions between apertures, is given by

$$S_1 = 1 - CQ_1/2\pi \sim (1+4ia/kd_1 d_2 \cos\theta)^{-1}. \tag{2.35}$$

A reciprocal argument may be used to infer that the far field at a distant point $\mathbf{x}(r,\theta)$ excited by a source at $\mathbf{x}'$ takes the form

$$\psi(\mathbf{x};\, \mathbf{x}') \sim S_1 F(\mathbf{x}')\, e^{ikr}/(-4\pi r) \tag{2.36}$$

together with incident and reflected waves.

It follows that a quadrupole source of strength T_{ij} situated near A induces a far field

$$\psi \sim S_1 T_{ij} \frac{\partial^2 F(\mathbf{x}')}{\partial x_i' \,\partial x_j'} \frac{e^{ikr}}{-4\pi r}, \tag{2.37}$$

and this is considerably greater than the incident and reflected waves, of order $k^2 e^{ikr}/r$, when ka is small.

The potential ψ near the (m,n)th aperture is obtained by multiplying ψ by $\exp(-ikd_1 m \sin\theta)$, and the field corresponding to a compact distribution of quadrupoles can be inferred by direct superposition. In particular, a compact region of turbulence near the apertures is acoustically equivalent to such a quadrupole distribution and has a scattered field that is much greater than the direct and specularly reflected fields.

3. Reflexion by a perforated screen with an infinite rigid backing plane

If the periodically perforated screen, whose geometry is that described in §2, has a parallel and rigid infinite plane backing at $z = -d_3$ (figure 2), full reflexion of an incident plane wave is to be expected; the adaptation of our prior analysis which suits these circumstances and brings out the changed aspect of reflexion is now undertaken.

We consider, as before, an obliquely incident plane wave such that

$$p_i = \exp\{-ik(z\cos\theta + x\sin\theta)\}$$

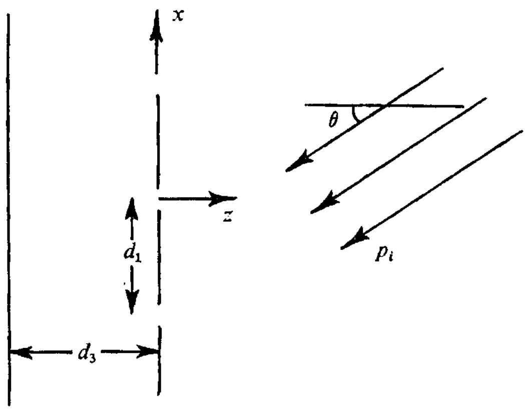

FIGURE 2. The geometry of the backed screen.

and again impose a periodicity requirement of the form

$$p(x+md_1, y+nd_2, z) = \exp[-ikd_1 m\sin\theta]\,p(x,y,z) \tag{3.1}$$

on the complete wave function. To recast the boundary-value problem in terms of an integral equation over a single aperture A centred at the origin, the total pressure fluctuation $p(x,y,z)$ on either side of the perforated screen is first expressed by means of its normal (i.e. z) derivative at this aperture; then, the requisite continuity of pressure at the plane of the aperture leads to the desired integral equation for $\partial p/\partial z$.

In the domain $z > 0$ an appropriate representation is

$$\begin{aligned} p(x,y,z) &= p_i(x,y,z)+p_i(x,y,-z) \\ &\quad + \sum_{m,n=-\infty}^{\infty}\int_{A_{mn}} G(x,y,z;\, x',y',0)\,\frac{\partial}{\partial z'}p(x',y',0)\,dx'\,dy' \\ &= p_i(x,y,z)+p_i(x,y,-z)+\int_A \mathscr{G}(x,y,z;\, x',y',0)\frac{\partial}{\partial z'}p(x',y',0)\,dx'\,dy', \\ &\qquad\qquad z>0, \end{aligned} \tag{3.2}$$

where G and $\mathscr{G}$ are given by (2.9) and (2.11), respectively. The asymptotic approximation (2.15) of $\mathscr{G}$ reveals that

$$p-p_i \sim R_2\exp\{-ik(-z\cos\theta+x\sin\theta)\} \quad \text{as} \quad z\to+\infty, \tag{3.3}$$

with

$$R_2 = 1-iQ_2/kd_1d_2\cos\theta, \tag{3.4}$$

where

$$Q_2 = \int_A \exp[ikx\sin\theta]\frac{\partial p}{\partial z}\,dx\,dy \equiv \int_A W(x,y)\,dx\,dy \tag{3.5}$$

is a constant to be determined.

In the region $-d_3 < z < 0$ between the screen and backing plane, we have

$$p(x,y,z) = -\sum_{m,n=-\infty}^{\infty}\int_{A_{mn}} G_1(x,y,z;\, x',y',0)\,\frac{\partial}{\partial z'}p(x',y',0)\,dx'\,dy', \tag{3.6}$$

where the Green's function G_1, whose z' derivative is zero on each of the planes $z' = 0$ and $z' = -d_3$, is fashioned by image summation of simple sources and takes the primitive form

$$G_1(x,y,z;\, x',y',0) = -\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\frac{\exp\{ik[(x-x')^2+(y-y')^2+(z-2ld_3)^2]^{\frac{1}{2}}\}}{[(x-x')^2+(y-y')^2+(z-2ld_3)^2]^{\frac{1}{2}}}. \tag{3.7}$$

Using the periodicity condition (3.1) and the definition

$$\mathcal{G}_1(x,y,z;\, x',y',0) = \sum_{m,n=-\infty}^{\infty} G_1(x,y,z;\, x'+md_1, y'+nd_2, 0)\exp[-ikd_1 m\sin\theta] \tag{3.8}$$

gives
$$p(x,y,z) = -\int_A \mathcal{G}_1(x,y,z;\, x',y',0)\frac{\partial p}{\partial z'}\,dx'\,dy', \quad -d_3 < z < 0, \tag{3.9}$$

whence the equivalence of (3.2) and (3.9) at all points of A yields the basic integral equation

$$\frac{1}{2}\int_A \{\mathcal{G}(x,y,0;x',y',0)+\mathcal{G}_1(x,y,0;x',y',0)\}$$
$$\times\exp[ik(x-x')\sin\theta]\,W(x',y')\,dx'\,dy' = -1, \quad (x,y) \text{ in } A, \tag{3.10}$$

for
$$W(x,y) = \exp[ikx\sin\theta]\,\partial p(x,y,0)/\partial z. \tag{3.11}$$

Evidently, (3.10) is similar to (2.14) and may be dealt with in a corresponding manner; thus, on availing ourselves of an estimate secured in appendix A, namely

$$\tfrac{1}{2}\{\mathcal{G}+\mathcal{G}_1\} = -(1/2\pi)(1/R+D)+O(1), \quad R^2 = (x-x')^2+(y-y')^2, \tag{3.12}$$

where

$$D = \frac{i\pi}{kd_1d_2\cos\theta} - \frac{\pi}{k^2d_1d_2d_3\cos^2\theta}, \tag{3.13}$$

the approximate version of the integral equation (3.10),

$$\frac{1}{2\pi}\int_A\left(\frac{1}{R}+D\right)W(x',y')\,dx'\,dy' = 1, \quad (x,y) \text{ in } A, \tag{3.14}$$

duplicates a previously encountered one, equation (2.21). This implies forthwith (through the replacement of C by D in (2.26) and (2.25)) that

$$Q_2 = \int_A W(x,y)\,dx\,dy = \frac{4a\pi}{\pi+2aD},$$

and
$$W(x,y) = (\tfrac{1}{2}\pi+aD)^{-1}(a^2-x^2-y^2)^{-\frac{1}{2}} \tag{3.15}$$

in the case of circular perforations. The reflexion coefficient supplied by (3.4) becomes, when use is made of (3.14),

$$R_2 = \frac{2a-k^2d_1d_2d_3\cos^2\theta+2iakd_3\cos\theta}{2a-k^2d_1d_2d_3\cos^2\theta-2iakd_3\cos\theta}, \tag{3.16}$$

and its validity is predicted for small values of $kd_>$ and $a/d_<$. Higher order corrections to (3.16), which depend on the arrangement of perforations, are left unspecified here, as well as the corrections for eccentricity of the apertures.

In keeping with our original expectations it is seen that $|R_2| = 1$ and that $R_2 \to 1$ when $d_3 \to 0$. On the other hand, if the wavenumber $k\cos\theta$ satisfies the 'resonance' condition

$$2a = (k\cos\theta)^2\,d_1d_2d_3 \tag{3.17}$$

for a cavity of volume $d_1d_2d_3$, then $R_2 = -1$ and the reflexion is like that of a perfectly *soft* screen.

Leaving aside incident and reflected waves, a previously used line of reasoning supplies the representation for pressure variation near A:

$$p \sim S_2 F(\mathbf{x}), \tag{3.18}$$

where $F(\mathbf{x})$ is the harmonic function specified by (2.33), and

$$S_2 = \left(1+\frac{2aD}{\pi}\right)^{-1} \sim \left(1+\frac{2ai}{kd_1d_2\cos\theta}-\frac{2a}{k^2d_1d_2d_3\cos^2\theta}\right)^{-1}, \tag{3.19}$$

from (3.15) and (3.13).

The far fields due to a monopole or quadrupole source near A are then given by formulae similar to (2.36) and (2.37), with S_1 replaced by S_2. In particular, if the wavenumber satisfies the resonance condition (3.17), then the factor S_2 has the large value

$$S_2 \sim -ikd_1d_2\cos\theta/2a = -i(d_1d_2/2a\,d_3)^{\frac{1}{2}}.$$

In the original plane-wave diffraction problem this condition implies that the local field, and wave amplitude between screen and backing plate, is large.

4. Reflexion by a periodic distribution of baffled pistons

A related problem discussed by Ffowcs Williams (1972) envisages aperture-filling rigid pistons that are constrained to move in the normal or z direction in accordance with an average impedance Z_0. The boundary condition at each piston, which may be regarded as a definition for the impedance, is (denoting the piston velocity by v)

$$-\frac{1}{\pi a^2}\int p\,dx\,dy = Z_0 v = \frac{Z_0}{ik\rho c}\frac{\partial p}{\partial z}, \tag{4.1}$$

and in terms of the function $\psi(x,y,z) = p(x,y,z)-p_i(x,y,z)-p_i(x,y,-z)$, we have

$$\frac{\partial\psi}{\partial z} = -K\int_{A_{mn}}\{2p_i(x,y,0)+\psi(x,y,0)\}\,dx\,dy, \quad \text{on} \quad A_{mn}, \tag{4.2}$$

where

$$K = ik\rho c/Z_0\pi a^2. \tag{4.3}$$

The boundary-value problem for ψ stated in §2 admits an exact solution when the aperture boundary condition (2.7) on the wave function itself is replaced by the specification (4.2) for its normal derivative. Let

$$F_i = \int_A p_i(x,y,0)\,dx\,dy = \pi a^2\frac{J_1(ka\sin\theta)}{\frac{1}{2}ka\sin\theta}, \tag{4.4}$$

and let

$$F = \int_A \psi(x,y,0)\,dx\,dy \tag{4.5}$$

designate a constant whose value is at present unknown; then it follows from inserting (4.2) into the representation

$$\psi(x,y,z) = \int_A \mathscr{G}(x,y,z;x',y',0)\frac{\partial}{\partial z'}\psi(x',y',0)\,dx'\,dy', \quad z \geqslant 0, \tag{4.6}$$

that

$$\psi(x,y,z) = -K(2F_i+F)\int_A \mathscr{G}(x,y,z;x',y',0)\,dx'\,dy', \quad z \geqslant 0, \tag{4.7}$$

and the magnitude of F is determined by integrating (4.7) over A. We thus obtain

$$F = -2KF_i I/(1+IK), \tag{4.8}$$

where
$$I = \int_A \mathscr{G}(x,y,0;\, x',y',0)\, dx\, dy\, dx'\, dy', \tag{4.9}$$

and the solution of the boundary-value problem, embodied in the wave function (4.7), is now complete.

Since the pressure far field assumes a plane-wave form

$$\psi(x,y,z) \sim \frac{2iKF_i^2}{kd_1 d_2 \cos\theta(1+IK)} \exp\{ik(z\cos\theta - x\sin\theta)\} \quad \text{as} \quad z \to \infty, \tag{4.10}$$

an exact expression for the reflexion coefficient R_3 may be obtained, viz.

$$R_3 = 1 - 2k\rho c F_i^2 / kd_1 d_2 \cos\theta(\pi a^2 Z_0 + ik\rho c I). \tag{4.11}$$

An asymptotic estimate for R_3, when $kd_> \ll 1$ and $a/d_< \ll 1$, follows readily from (4.11) on substitution of those appropriate to F_i and I, which are

$$F_i = \pi a^2[1+O(ka)^2]$$

and
$$\begin{aligned} I &= -\frac{1}{2\pi}\int_A \left(\frac{1}{R}+C\right) dx\, dy\, dx'\, dy' + O(a^3 a/d_<) \\ &= -(1/2\pi)\left(\tfrac{16}{3}\pi a^3 + C\pi^2 a^4\right) + O(a^3 a/d_<), \end{aligned}$$

respectively. If the reflexion coefficient is written as

$$R_3 = \frac{Z\cos\theta - \rho c}{Z\cos\theta + \rho c}(1+O(ka)^2)$$

the effective impedance Z of the plane with baffled pistons is

$$Z = \frac{d_1 d_2}{\pi a^2}\left(Z_0 - i\frac{8}{3\pi}ka\rho c\right) - i\frac{d_1 d_2}{4} k\rho c F(d_1, d_2),$$

where the last term contains the structure factor F specified in (2.28) and furnishes a small correction, when $a/d_< \ll 1$, to the result obtained by Ffowcs Williams (1972) for arbitrarily spaced pistons.

Appendix A. The Green's functions and their approximations

The function $\mathscr{G}(x,y,z;\, x',y',0)$ defined by

$$\mathscr{G}(x,y,z;\, x',y',0) = -\frac{1}{2\pi}\sum_{m,n=-\infty}^{\infty} \frac{\exp\{ikR(x,y,z;\, x'+md_1, y'+nd_2, 0)\}}{R(x,y,z;\, x'+md_1, y'+nd_2, 0)} \exp[-ikd_1 m\sin\theta], \tag{A1}$$

where
$$R^2(x,y,z;\, x',y',0) = (x-x')^2 + (y-y')^2 + z^2, \tag{A2}$$

can, through a two-fold use of the Poisson summation formula, be recast in the alternative form

$$\mathscr{G}(x,y,z;\, x',y',0) = -\frac{1}{\pi d_1 d_2}\int_{-\infty}^{\infty} \sum_{\nu,\mu=-\infty}^{\infty} \frac{\exp\{i(x-x')(-k\sin\theta + 2\pi\nu/d_1) + i(y-y')\, 2\pi\mu/d_2 + i\zeta z\}}{(k\sin\theta - 2\pi\nu/d_1)^2 + (2\pi\mu/d_2)^2 + \zeta^2 - k^2}\, d\zeta, \tag{A3}$$

with a path of integration that is indented above the pole on the negative real axis and below the pole on the positive real axis.

For large positive values of z, it is convenient to deform the path of integration into the upper half of the ζ plane and to express the integral as a sum of residues. There is a contribution from one pole at $\zeta = k\cos\theta$ on the real axis, corresponding to $\nu = \mu = 0$; the other values of ν and μ give rise to poles on the imaginary axis and provide residues that are exponentially small when z is large, with kd_1 and $kd_2 \ll 1$. Thus the only propagating mode is characterized by $\nu = \mu = 0$ and

$$\mathscr{G}(x,y,z;\, x',y',0) \sim -i\exp\{ik(z\cos\theta+(x-x')\sin\theta)\}/kd_1 d_2\cos\theta \quad \text{as} \quad z \to +\infty. \tag{A 4}$$

We require also an asymptotic estimate, for small values of $kd_>$ and $a/d_<$, of the function

$$\mathscr{G}(x,y,0;x',y',0) = -\frac{\exp\{ikR(x,y;x',y')\}}{2\pi R(x,y;x',y')} - \frac{1}{2\pi}\sum_{m,n=-\infty}^{\infty}{}' \frac{\exp\{ikR(x,y;x'+md_1,y'+nd_2)\}}{R(x,y;x'+md_1,y'+nd_2)} \times \exp[-ikd_1 m\sin\theta], \tag{A 5}$$

where $$R^2(x,y;\, x',y') = (x-x')^2+(y-y')^2$$

and the symbol Σ' denotes a double sum exclusive of $(m,n) = (0,0)$, with both the points (x,y) and (x',y') lying in a circular aperture A. Since $ka \ll 1$ and $a \ll d_<$, it follows that the first term of (A 5) may be replaced by $-1/2\pi R$ and that the particular values $x = x' = y = y' = 0$ may be assigned in the double sum; thus

$$\mathscr{G}(x,y,0;x',y',0) = -\frac{1}{2\pi R}+A_1+A_2+o(1) \quad \text{as} \quad ka \to 0 \tag{A 6}$$

with $$A_1 = -\frac{1}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{ik|nd_2|}}{|nd_2|} \tag{A 7}$$

and $$A_2 = -\frac{1}{2\pi}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}\sum_{n=-\infty}^{\infty} \frac{\exp\{ik(m^2d_1^2+n^2d_2^2)^{\frac{1}{2}}\}}{(m^2d_1^2+n^2d_2^2)^{\frac{1}{2}}}\exp[-ikmd_1\sin\theta]. \tag{A 8}$$

The explicit evaluation of (A 7) yields

$$\begin{aligned} A_1 &= (\pi d_2)^{-1}\log(1-e^{ikd_2}) \\ &\sim (\pi d_2)^{-1}\{\log(kd_2)-\tfrac{1}{2}i\pi\} \quad \text{as} \quad kd_2 \to 0 \end{aligned} \tag{A 9}$$

and as far as A_2 is concerned, we find it convenient to identify the sum over all n with the wave function at $(md_1, 0, 0)$ arising from point sources on the y axis at $y = 0, \pm d_2, \pm 2d_2, \ldots$. An aggregate of this type comprises the image system appropriate to a simple source at the origin in a parallel-sided duct with rigid plane boundaries at $y = \pm\frac{1}{2}d_2$. Hence the sum

$$\bar{G}(md_1,0,0) = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\exp\{ik(m^2d_1^2+n^2d_2^2)^{\frac{1}{2}}\}}{(m^2d_1^2+n^2d_2^2)^{\frac{1}{2}}} \tag{A 10}$$

represents a specific value of the Green's function $\bar{G}(x,y,z)$ for the duct configuration, which satisfies the equations

$$\left.\begin{aligned}(\nabla^2+k^2)\,\bar{G} &= 2\delta(x)\,\delta(y)\,\delta(z),\\ \partial\bar{G}/\partial y = 0 \quad &\text{at} \quad y = \pm\tfrac{1}{2}d_2.\end{aligned}\right\} \qquad \text{(A 11)}$$

and a radiation condition at infinity.

The solution of such an axially symmetric waveguide problem can be achieved in elementary fashion, on separating variables in a cylindrical polar co-ordinate system (r,y), $r^2 = x^2+z^2$, and it is found that

$$\bar{G}(x,0,0) = \frac{1}{2id_2}H_0^{(1)}(k|x|) - \frac{2}{\pi d_2}\sum_{n=1}^{\infty}K_0\left\{\left(\frac{4n^2\pi^2}{d_2^2}-k^2\right)^{\frac{1}{2}}|x|\right\}$$

with the conventional designations for a Hankel function $H_0^{(1)}$ of the first kind and a modified Bessel function K_0. From (A 8) and (A 10) we next obtain

$$\begin{aligned}A_2 = &-\frac{i}{d_2}\sum_{m=1}^{\infty}H_0^{(1)}(mkd_1)\cos(mkd_1\sin\theta)\\ &-\frac{4}{\pi d_2}\sum_{m,n=1}^{\infty}K_0\left\{\left(\frac{4n^2\pi^2}{d_2^2}-k^2\right)^{\frac{1}{2}}md_1\right\}\cos(mkd_1\sin\theta)\\ \sim &-\frac{i}{d_2}\sum_{m=1}^{\infty}H_0^{(1)}(mkd_1)\cos(mkd_1\sin\theta) - \frac{4}{\pi d_2}\sum_{m,n=1}^{\infty}K_0\left(2mn\pi\frac{d_1}{d_2}\right)\\ &\text{as} \quad kd_1, kd_2 \to 0. \end{aligned} \qquad \text{(A 12)}$$

Since the sum of Hankel functions has the alternative form (cf. Ryzhik & Gradshteyn 1965, §8.522)

$$\begin{aligned}\sum_{m=1}^{\infty}H_0^{(1)}(mkd_1)\cos(mkd_1\sin\theta) = &-\frac{i}{\pi}\left\{\gamma+\log\frac{kd_1}{4\pi}\right\}-\frac{1}{2}+\frac{1}{kd_1\cos\theta}\\ -i\sum_{m=1}^{\infty}\left[\frac{1}{\{(2m\pi-kd_1\sin\theta)^2-k^2d_1^2\}^{\frac{1}{2}}}-\frac{1}{2m\pi}+\frac{1}{\{(2m\pi+kd_1\sin\theta)^2-k^2d_1^2\}^{\frac{1}{2}}}-\frac{1}{2m\pi}\right]\\ \sim &-\frac{i}{\pi}\left\{\gamma+\log\frac{kd_1}{4\pi}\right\}-\frac{1}{2}+\frac{1}{kd_1\cos\theta} \quad \text{as} \quad kd_1\to 0,\end{aligned} \qquad \text{(A 13)}$$

where $\gamma = 0{\cdot}5771\ldots$ is the Euler constant, we deduce, after collecting together the results (A 6)–(A 13), that

$$\mathscr{G}(x,y,0;x',y',0) = -\frac{1}{2\pi R}-\frac{i}{kd_1d_2\cos\theta}-\tfrac{1}{4}F(d_1,d_2)+o(1) \qquad \text{(A 14)}$$

as $kd_>$ and $a/d_< \to 0$, with

$$F(d_1,d_2) = \frac{4}{\pi d_2}\left[\gamma+\log\left(\frac{d_1}{4\pi d_2}\right)+4\sum_{m,n=1}^{\infty}K_0\left(2mn\pi\frac{d_1}{d_2}\right)\right]. \qquad \text{(A 15)}$$

As the structure factor F is independent of the angle of incidence θ and the Green's function $\mathscr{G}$ is obviously symmetric with respect to the spacing parameters d_1 and d_2 when $\theta = 0$, it follows at once that

$$F(d_1,d_2) = F(d_2,d_1). \qquad \text{(A 16)}$$

The double sum in (A 15) is rapidly convergent when $d_1 > d_2$ and for $d_1 < d_2$ it is advisable to exploit the symmetry property (A 16) by interchanging the parameters.

A related Green's function $\mathscr{G}_1(x, y, z;\, x', y', z')$, which presents itself (§4) if the perforated screen has a backing plane, takes the form of a triple sum, in accordance with (4.7) and (4.8), namely

$$\mathscr{G}_1(x,y,z;x',y',0) = -\frac{1}{2\pi}\sum_{l,m,n=-\infty}^{\infty}\frac{\exp\{ikR_-(x,y,z;x'+md_1,y'+nd_2,2ld_3)\}}{R_-(x,y,z;x'+md_1,y'+nd_2,2ld_3)}\exp[-ikd_1 m\sin\theta], \tag{A 17}$$

where $$R_-^2(x,y,z;\, x',y',z') = (x-x')^2+(y-y')^2+(z-z')^2. \tag{A 18}$$

The procedure just described in connexion with $\mathscr{G}(x,y,z;\, x',y',0)$ also enables us to obtain an asymptotic estimate for $\mathscr{G}_1(x,y,0;x',y',0)$ when $kd_>$ and $a/d_<$ are both small. Thus, we begin by isolating the $l = m = n = 0$ term in (A 17) and thereafter substituting for the exact representation

$$\mathscr{G}_1(x,y,0;x',y',0) = -\frac{\exp\{ikR_-(x,y,0;\, x',y',0)\}}{2\pi R_-(x,y,0;\, x',y',0)} - \frac{1}{2\pi}\sum_{\substack{l,m,n=-\infty\\(l,m,n)\neq(0,0,0)}}^{\infty}\frac{\exp\{ikR_-(x,y,0;x'+md_1,y'+nd_2,2ld_3)\}}{R_-(x,y,0;\, x'+md_1,y'+nd_2,2ld_3)}\exp[-ikd_1 m\sin\theta]$$

an approximate form

$$\mathscr{G}_1(x,y,0;\, x',y',0) \sim -\frac{1}{2\pi R} - \frac{1}{2\pi}\sum_{\substack{l,m,n=-\infty\\(l,m,n)\neq(0,0,0)}}^{\infty}\frac{\exp\{ikR_-(0,0,0;md_1,nd_2,2ld_3)\}}{R_-(0,0,0;md_1,nd_2,2ld_3)}\exp[-ikd_1 m\sin\theta],$$

which rests on the premise that ka and $a/d_{1,2}$ remain small compared with unity. A rearrangement of the triple sum, giving

$$\mathscr{G}_1(x,y,0;\, x',y',0) \sim -1/2\pi R + B_1 + B_2, \tag{A 19}$$

where $$B_1 = -\frac{1}{2\pi}\sum_{l,n=-\infty}^{\infty}\frac{\exp\{ik(n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}\}}{(n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}} \tag{A 20}$$

and

$$B_2 = -\frac{1}{2\pi}\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\sum_{l,n=-\infty}^{\infty}\frac{\exp\{ik(m^2d_1^2+n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}\}}{(m^2d_1^2+n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}}\exp[-ikd_1 m\sin\theta], \tag{A 21}$$

enables the desired characterization of B_1 to be directly inferred from that previously given for $A_1 + A_2$, on replacing d_2 with $2d_3$ and setting $\theta = 0$. Accordingly,

$$B_1 = -i/(2kd_2d_3) + O(1) \tag{A 22}$$

when $kd_{2,3} \ll 1$, and there remains only the determination of an analogous estimate for B_2; adhering to the prior style of calculation, the summand of (A 21) is regarded as a wave function at $(md_1, 0, 0)$ with source at $(0, nd_2, 2ld_3)$ and the double sum over l and n is viewed in terms of an image array which corresponds to a simple source at the origin in the presence of rigid walls at

$$y = \pm\tfrac{1}{2}d_2, \quad z = \pm d_3.$$

It follows that the sum

$$F_1(md_1, 0, 0) = -\frac{1}{2\pi} \sum_{l,n=-\infty}^{\infty} \frac{\exp\{ik(m^2d_1^2+n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}\}}{(m^2d_1^2+n^2d_2^2+4l^2d_3^2)^{\frac{1}{2}}} \tag{A 23}$$

may be found after particular arguments are adopted in the Green's function $F_1(x, y, z)$ of a rectangular duct, specified by the differential equation

$$(\nabla^2+k^2)\,F_1 = 2\delta(x)\,\delta(y)\,\delta(z)$$

and the boundary conditions

$$\partial F_1/\partial y = 0 \quad \text{when} \quad y = \pm\tfrac{1}{2}d_2,$$
$$\partial F_1/\partial z = 0 \quad \text{when} \quad z = \pm d_3.$$

A representation of F_1, arrived at through separation of variables, yields

$$F_1(md_1, 0, 0) = \frac{e^{ik|m|d_1}}{2ikd_2d_3} - \frac{1}{2d_2d_3} \sum_{\substack{l,n=-\infty \\ (l,n)\neq(0,0)}}^{\infty} \frac{\exp\{-(4n^2\pi^2/d_2^2+l^2\pi^2/d_3^2-k^2)^{\frac{1}{2}}\,|m|d_1\}}{(4n^2\pi^2/d_2^2+l^2\pi^2/d_3^2-k^2)^{\frac{1}{2}}} \tag{A 24}$$

and thus, on application of (A 21), (A 23) and (A 24), it follows, in the limit $k \to 0$, that

$$B_2 \sim \frac{1}{2ikd_2d_3} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \exp[ik|m|d_1 - ikd_1m\sin\theta] + O(1)$$

$$= \frac{1}{k^2d_1d_2d_3\cos^2\theta} - \frac{1}{2ikd_2d_3} + O(1) \tag{A 25}$$

when $kd_>$ is small. Finally, (A 19), (A 22) and (A 25) combine to reveal that

$$\mathscr{G}_1(x, y, 0;\, x', y', 0) = -\frac{1}{2\pi R} + \frac{1}{k^2d_1d_2d_3\cos^2\theta} + O(1). \tag{A 26}$$

Appendix B. Solution of an integral equation by expansion of its kernel

The integral equation on a plane domain of finite area A

$$\frac{1}{2\pi}\int_A (1/R)f(x', y')\,dx'\,dy' = g(x, y), \quad R^2 = (x-x')^2+(y-y')^2, \tag{B 1}$$

which for $g =$ constant has previously been found to be relevant to scattering from a perforated screen, also characterizes the electrostatic charge density $f(x, y)$ on a plane disk A at which the potential takes the assigned value $g(x, y)$. For the circular disk, aptly described with plane polar co-ordinates, methods of solving the equation have been presented some time ago (Serini 1923; Sbrana 1924)† and rediscovered more recently (Copson 1947); they rely on special integral representations of the kernel and thereby restate the problem in simpler and more manageable terms, requiring only the known solutions of a pair of one-variable Abel-type integral equations. In the following, we shall develop an alternative

† It may be noted that these references are absent from the extensive bibliography compiled by Sneddon (1966).

technique that is applicable to an elliptical domain A $(x^2/a^2+y^2/b^2 = 1)$ as well, and thus surpasses in scope the ones hitherto available.

New variables (θ, ϕ) are introduced by the transformation

$$x = a\sin\theta\cos\phi, \quad y = b\sin\theta\sin\phi \tag{B 2}$$

so that the domain A is given by $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, $0 \leqslant \phi \leqslant 2\pi$, and the integral equation (B 1) takes the form

$$\frac{ab}{2\pi}\int_0^{\frac{1}{2}\pi} d\theta' \int_0^{2\pi} d\phi' (1/R) f(\theta',\phi') \sin\theta' \cos\theta' = g(\theta,\phi), \tag{B 3}$$

for (θ, ϕ) in the domain A, where

$$R^2 = a^2(\sin\theta\cos\phi - \sin\theta'\cos\phi')^2 + b^2(\sin\theta\sin\phi - \sin\theta'\sin\phi')^2. \tag{B 4}$$

In an obvious notation $f(\theta,\phi)$ and $g(\theta,\phi)$ are used to denote $f(x,y)$ and $g(x,y)$ written in terms of θ and ϕ.

It is expedient to extend the domain of integration to include all θ from 0 to π, by defining a new unknown function

$$F(\theta,\phi) = \begin{Bmatrix} \cos\theta f(\theta,\phi) & \text{for} \quad 0 \leqslant \theta \leqslant \frac{1}{2}\pi, \\ F(\pi-\theta,\phi) & \text{for} \quad \frac{1}{2}\pi \leqslant \theta \leqslant \pi. \end{Bmatrix} \tag{B 5}$$

Similarly the region of definition of $g(\theta,\phi)$ is extended by defining

$$g(\theta;\phi) = g(\pi-\theta,\phi) \quad \text{for} \quad \tfrac{1}{2}\pi \leqslant \theta \leqslant \pi. \tag{B 6}$$

Thus $F(\theta,\phi)$ and $g(\theta,\phi)$ are even functions of $\theta-\frac{1}{2}\pi$. Now it is readily seen from (B 4) that the kernel function $1/R$ is an even function of both $\theta-\frac{1}{2}\pi$ and $\theta'-\frac{1}{2}\pi$, and this ensures that the integral equation (B 3) can be written as

$$\frac{ab}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' (1/R) \sin\theta' F(\theta',\phi') = g(\theta,\phi) \tag{B 7}$$

for all (θ,ϕ) on the extended domain D; $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$. An advantage gained from the formulation of the integral equation (B 7) on the domain D is that its kernel $1/R$ can be expressed conveniently as a series expansion of functions with certain orthogonality properties in D, as is now shown.

The double Fourier integral

$$\frac{1}{R} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\{i\zeta(x-x')+i\eta(y-y')\}}{(\zeta^2+\eta^2)^{\frac{1}{2}}} d\zeta d\eta \tag{B 8}$$

provides a suitable form with which to begin the analysis. The transformation

$$\zeta = (\rho/a)\cos\Phi, \quad \eta = (\rho/b)\sin\Phi, \tag{B 9}$$

together with (B 2) leads to the result

$$\frac{1}{R} = \frac{1}{4\pi(ab)^{\frac{1}{2}}}\int_0^{2\pi} \frac{d\Phi}{[(b/a)\cos^2\Phi + (a/b)\sin^2\Phi]^{\frac{1}{2}}} \times \int_{-\infty}^{\infty} d\rho \exp\{i\rho[\sin\theta\cos(\phi-\Phi) - \sin\theta'\cos(\phi'-\Phi)]\}. \tag{B 10}$$

The exponential factors in (B 10) are next given individual expansions which stem from the general and classical result

$$\exp(i\rho\cos\Theta) = \exp\{i\rho(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\gamma)\}$$

$$= \sum_{l=0}^{\infty}(2l+1)\,i^l j_l(\rho)\,P_l(\cos\Theta), \tag{B 11}$$

where $j_l(x)$ is the spherical Bessel function

$$j_l(x) = (\pi/2x)^{\frac{1}{2}}\,J_{l+\frac{1}{2}}(x), \tag{B 12}$$

and $P_l(\cos\Theta)$ is the Legendre polynomial whose addition theorem has the form

$$P_l(\cos\Theta) = \sum_{m=-l}^{m=l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\alpha)\,P_l^m(\cos\beta)\,e^{im\gamma}. \tag{B 13}$$

It is convenient to work with the renormalized Legendre functions

$$\bar{P}_l^m(z) = (l+\tfrac{1}{2})^{\frac{1}{2}}\{(l-m)!/(l+m)!\}^{\frac{1}{2}}\,P_l^m(z), \tag{B 14}$$

which satisfy the orthogonality conditions

$$\int_0^{\pi}\sin\theta\bar{P}_l^m(\cos\theta)\,\bar{P}_l^{m'}(\cos\theta)\,d\theta = \delta_{mm'}, \tag{B 15}$$

where $\delta_{mm'}$ is the Kronecker delta.

The spherical Bessel functions j_l of formula (B 12) have the orthogonality property

$$\int_{-\infty}^{\infty}j_l(\rho)j_{l'}(\rho)\,d\rho = \pi(2l+1)^{-1}\,\delta_{ll'}. \tag{B 16}$$

On substituting (B 13) and (B 14) into (B 11) and setting $\alpha = \theta$, $\beta = \frac{1}{2}\pi$ and $\gamma = \phi-\Phi$, it is seen that the first exponential factor in the integral (B 10) may be written as

$$\exp\{i\rho\sin\theta\cos(\phi-\Phi)\} = 2\sum_{l=0}^{\infty}\sum_{m=-l} i^l j_l(\rho)\,\bar{P}_l^m(0)\,\bar{P}_l^m(\cos\theta)\,e^{im(\phi-\Phi)}, \tag{B 17}$$

and by complex conjugation the second exponential factor is

$$\exp\{-i\rho\sin\theta'\cos(\phi'-\Phi)\} = 2\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} i^{-l'}j_{l'}(\rho)\,\bar{P}_{l'}^{m'}(0)\,\bar{P}_{l'}^{m'}(\cos\theta')\,e^{-im'(\phi'-\Phi)}. \tag{B 18}$$

When (B 17) and (B 18) are substituted into the integral (B 10) and use is made of the orthogonality properties (B 15) and (B 16), it is found that $1/R$ has the representation

$$\frac{1}{R} = (ab)^{-\frac{1}{2}}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{m'=-l}^{l}C(l,m,m')\,\bar{P}_l^m(\cos\theta)\,\bar{P}_l^{m'}(\cos\theta')\,e^{im\phi-im'\phi'}, \tag{B 19}$$

where

$$C(l,m,m') = \frac{\bar{P}_l^m(0)\,\bar{P}_l^{m'}(0)}{2l+1}\int_0^{2\pi}\left(\frac{b}{a}\cos^2\Phi+\frac{a}{b}\sin^2\Phi\right)^{-\frac{1}{2}}e^{i(m'-m)\Phi}\,d\Phi. \tag{B 20}$$

It is noted here that $C(l, m, m')$ is zero if either $l-m$ or $l-m'$ is odd, since the respective functions $\bar{P}_l^m(0)$ and $\bar{P}_l^{m'}(0)$ (B 20) are zero under these conditions. Thus the sums with respect to m and m' in (B 20) are taken only over even values of $l-m$ and $l-m'$; this reflects that fact that $1/R$ is an even function of both $\theta-\frac{1}{2}\pi$ and $\theta'-\frac{1}{2}\pi$.

The series expansion (B 19) for $1/R$ now leads to a solution of the integral equation (B 7). Guided by (B 19), we write $F(\theta,\phi)$ and $g(\theta,\phi)$ in the similar forms

$$F(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m-l \text{ even}}}^{l} a_{lm} \bar{P}_l^m(\cos\theta)\, e^{im\phi} \tag{B 21}$$

and

$$g(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m-l \text{ even}}}^{l} g_{lm} \bar{P}_l^m(\cos\theta)\, e^{im\phi}, \tag{B 22}$$

where the g_{lm} are known in principle since $g(\theta,\phi)$ is known, and the coefficients a_{lm} are to be found. The restriction to even values of $m-l$ in (B 21) and (B 22) results from the fact that F and g are even functions of $\theta-\frac{1}{2}\pi$. Substitution of (B 21) and (B 22) into the integral equation (B 7) and use of the orthogonality relation (B 15) leads to the result

$$\tfrac{1}{2}(ab)^{\frac{1}{2}} \sum_{\substack{m_1=-l \\ m_1-l \text{ even}}}^{l} a_{lm_1} C(l, m, m_1) = g_{lm} \quad (-l \leqslant m \leqslant l). \tag{B 23}$$

For each fixed value of l, the system (B 23) is an $(l+1)\times(l+1)$ system of equations for the constants $a_{l,-l}, a_{l,-l+2}, \ldots, a_{l,l}$.

The case of particular interest in this work is that in which g = constant = α. Thus the expansion (B 22) has

$$g_{00} = \alpha, \quad g_{lm} = 0 \quad \text{for} \quad (l, m) \neq (0, 0), \tag{B 24}$$

and the system (B 23) reduces to the trivial one

$$\tfrac{1}{2}(ab)^{\frac{1}{2}} a_{00} C(0, 0, 0) = \alpha, \tag{B 25}$$

with $a_{lm} = 0$ for $(l, m) \neq (0, 0)$. Now $C(0, 0, 0)$ is easily shown from its definition (B 20) to have the value $2(b/a)^{\frac{1}{2}} K(\epsilon)$, where K denotes an elliptic integral and $\epsilon = (1-b^2/a^2)^{\frac{1}{2}}$. Formulae (B 25), (B 21) and (B 5) show that the solution of the integral equation (B 1) is given by

$$f = \frac{\alpha}{bK(\epsilon)} \sec\theta = \frac{\alpha}{bK(\epsilon)} \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}, \tag{B 26}$$

as previously quoted in equation (2.24).

Another case worthy of special mention is that in which g is a polynomial of degree M in x and y; thus g is a polynomial in $\cos\phi$ and $\sin\phi$, and the sum (B 22) for g involves values of m absolutely bounded by M. Since the coefficients a_{lm} of the system (B 23) must also have $|m| < M$ it follows that $F(\theta,\phi) = \cos\theta f(\theta,\phi)$ is also a polynomial $P_M(x, y)$ of degree M. Thus

$$f = P_M(x,y) \sec\theta = P_M(x,y)\,(1-x^2/a^2-y^2/b^2)^{-\frac{1}{2}},$$

and this is a simple proof of the theorem of Galin that is referred to by Sneddon (1966).

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